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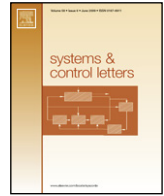
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Stability and synchronization preserving model reduction of multi-agent systems

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ABSTRACT

In this paper, stability and synchronization preserving model reduction schemes are developed for linear multi-agent systems. The multi-agent systems that are considered here are composed of general, yet identical linear subsystems, and the communication topology is assumed to be time-independent. First, under the assumption that the agents have stable internal dynamics and the network is stable, the dynamic order of the agents is reduced such that the corresponding reduced order network is again stable. Then, starting from a synchronized network where agents are allowed to have unstable dynamics, a reduced order model for the network which preserves synchronization is obtained. In addition, model reduction error bounds are established to compare the behavior of the original network to that of the reduced order model. The proposed results are illustrated through a numerical example.

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1. Introduction

The problems of consensus, coordination, and synchronization of multi-agent systems have received compelling attention in the last decade. A multi-agent system is a collection of systems (agents) that interact to fulfil a certain task. The behavior of a multi-agent system, hence, is determined by both the dynamics of the agents and the communication topology of the network which specifies admissible communication among the agents.

An important issue in the context of multi-agent system is consensus. Consensus roughly means that the agents agree on a certain quantity of interest. The focus of pioneering works in this direction has been on communication constraints concerning connectivity, time-varying topologies and time delays (see e.g. [1–4]). Consequently, the dynamics of the individual agents has been somewhat ignored as the focus has been mainly on the case of simple or double integrators.

In the last few years, however, attention has shifted to analysis and design of multi-agent systems with general linear dynamics. One of the most popular frameworks that have emerged in this direction studies multi-agent systems which are composed of several copies of interacting identical linear input–output subsystems. In this rather general framework, the term “consensus” has been replaced by “synchronization” in order to put emphasis on the dynamics of the agents. In the context of synchronization, similar to

consensus, the goal of communication is to achieve a common solution of the individual agents' dynamics. Among the numerous instances of available research in this direction, we refer to [5,6], or [7]. In this paper, similarly, we consider a network of agents with general, yet identical, linear dynamics, and we assume that partial information of the agents is transmitted via network communication. Thus, the structure we consider here also captures the case where the agents are coupled through a general static state-feedback or through an output–feedback protocol.

In the context of linear time-invariant systems, the *complexity* of a system is in general measured by its dynamic order, i.e., the number of state components in a state space representation of the system. Trying to reduce the complexity of models has led to the development of various model reduction techniques over the last decades. Perhaps the most well-known of these is Lyapunov balanced truncation (see [8–10]). In this approach, first the system is transformed into a so-called balanced form, and next a reduced order model is obtained by truncation. Other types of balancing include stochastic, bounded real, and positive real balancing (see [11–13]).

Since the multi-agent systems we consider here are composed of several linear time-invariant systems interconnected by a time-independent topology, they can be represented, as a whole, by a finite-dimensional linear time-invariant system. The dynamic order of such a representation for a network with p agents is in general p times that of the individual agents. As a result, the complexity of the network model will be reduced substantially by reducing the dynamic order of the agents, especially in large-scale networks. This motivates us to exploit available model reduction techniques to obtain a simpler, lower order model for the network which approximates the behavior of the original one.

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A critical issue in model reduction is preservation of qualitative properties of the original model in the reduced order model. For instance, stability, contractivity and passivity are preserved in the reduced order model obtained by Lyapunov balancing, bounded real and positive real balancing, respectively. In this paper, we consider, separately, stability of the network and synchronization of the network as desired qualitative properties to be preserved in the reduced order network. Preservation of synchronization is particularly challenging as agents typically have integrators and unstable dynamics in the context of networks and multi-agent systems.

There are several factors which make this problem non-trivial and challenging, such as:

1. Although a multi-agent system as a whole can be represented as a finite-dimensional linear time-invariant system, this representation, however, has a certain structure imposed by the network communication topology. This structure, of course, must be preserved in the reduced order network, and, therefore, we deal with a kind of structure preserving model reduction problem. Besides, recall that the multi-agent systems we consider here are composed of linear agents which are identical. Thus, a direct application of available model reduction techniques may introduce heterogeneity into the network which adds to the complexity of the model, despite the fact that the dynamic order of the network might have been reduced.
2. Most of the well-established model reduction techniques rely on the assumption of stability of the system. However, as mentioned, agents typically have integrators and unstable dynamics in the context of synchronization. Therefore, in this case, popular model reduction methods like Lyapunov balancing, bounded real, or positive real balancing are not directly applicable to the dynamics of the individual agents. Moreover, for linear time-invariant systems, *closeness* of the reduced order model and the original one can be estimated by the difference in \mathcal{H}_∞ -norm of the corresponding transfer matrices. However, this is not readily applicable to the individual agents of the network due to the presence of possible unstable dynamics, which hinders measuring how well the reduced order network approximates the original one.

The structure of the paper is as follows. In Section 2, some notations and basic material needed in the sequel are provided. In Section 3, a stability preserving model reduction scheme for multi-agent systems is proposed. A synchronization preserving model reduction approach is established in Section 4, and is applied to a numerical example in Section 5. The paper ends with conclusions in Section 6.

2. Preliminaries

2.1. Multi-agent systems

Let $G = (V, E)$ be an undirected (unweighted) graph where $V = \{1, 2, \dots, p\}$ is the vertex set and $E \subseteq V \times V$ is the edge set. A diffusively coupled multi-agent system consists of a collection of identical linear input/state/output systems given by

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + Bu_i(t) \\ y_i(t) &= Cx_i(t), \end{aligned} \quad (1a)$$

together with the diffusive coupling rule

$$u_i(t) = - \sum_{(i,j) \in E} (y_i(t) - y_j(t)), \quad (1b)$$

where $i \in \{1, 2, \dots, p\}$, $x_i \in \mathbb{R}^n$ is the state of agent i , and $u_i \in \mathbb{R}^m$ is the diffusive coupling term. Throughout this paper, it is assumed that the state space representation (1a) is minimal. Let L denote the

Laplacian matrix corresponding to the graph $G = (V, E)$. Then, the multi-agent system (1) can be written in compact form as

$$\dot{x}(t) = \mathcal{A}x(t) \quad (2)$$

where $x = \text{col}(x_1, x_2, \dots, x_p)$, $\mathcal{A} := I_p \otimes A - L \otimes BC$, and “ \otimes ” denotes the Kronecker product.

Note that the Laplacian matrix always has an eigenvalue at zero, and the multiplicity of this zero eigenvalue is associated with the connectedness of G . In fact, $G = (V, E)$ is connected if and only if the multiplicity of the zero eigenvalue of the Laplacian matrix is 1 (see e.g. [14, p. 27]). In Section 3, we do not assume the connectedness of G , while in Section 4 the graph G is assumed to be connected.

Next to the Laplacian matrix, we will use another matrix associated with the graph G , the so-called *incidence matrix* of a graph. After the edges are labeled and oriented arbitrarily, the incidence matrix of G , denoted by R , is defined as (see [14, p. 21]):

$$R_{ij} = \begin{cases} 1 & \text{if vertex } i \text{ is the head of edge } j \\ -1 & \text{if vertex } i \text{ is the tail of edge } j \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

for $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, q$, where p and q are the total number of vertices and edges, respectively. The relationship between the incidence matrix and the Laplacian matrix is captured by the following equality:

$$L = RR^\top. \quad (4)$$

2.2. Model reduction

In this subsection, we review some basic material and facts on model reduction by balanced truncation. Consider the finite dimensional, linear time-invariant system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du, \end{aligned} \quad (5)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{w \times n}$, and $D \in \mathbb{R}^{w \times m}$. Assume that the matrix A is Hurwitz, and the state space representation (5) is minimal. We shortly denote this system by $\Gamma(A, B, C, D)$, and we use the notation $\Gamma(A, B, C)$ for the case where $D = 0$. In general, the model reduction by balanced truncation consists of two major steps, namely balancing and truncation. Balancing is, basically, finding a nonsingular state space transformation T that diagonalizes appropriately chosen positive definite matrices P and Q in a covariant and contravariant manner, respectively. This means that P transforms to TPT^\top and Q transforms to $T^{-\top}QT^{-1}$, and the transformed matrices should be diagonal and equal. Notable types of balancing are Lyapunov balancing, bounded real (BR) balancing, and positive real (PR) balancing. In Lyapunov balancing the matrices P and Q are chosen to be the reachability and observability gramians, which are obtained from the following Lyapunov equations:

$$\begin{aligned} A^\top Q + QA + C^\top C &= 0 \\ AP + PA^\top + BB^\top &= 0. \end{aligned}$$

A reduced order model can be obtained by balancing the pair of positive definite matrices (P, Q) , and truncating the state components which are of least importance; in other words, the states which are relatively difficult to reach and observe. Let G and G_r denote the transfer matrices of the original and the reduced order model, respectively. Then the model reduction error given by $\|G - G_r\|_\infty$ is bounded from above by twice the sum of the neglected Hankel singular values (HSV). For details, we refer to [15].

Instead of using the Lyapunov equations above, one can also work with solutions of Lyapunov inequalities, and obtain a reduced

order model based on the so-called generalized gramians. More precisely, let Q_g and P_g be positive definite solutions of the inequalities

$$A^\top Q_g + Q_g A + C^\top C \leq 0 \quad (6a)$$

$$AP_g + P_g A^\top + BB^\top \leq 0. \quad (6b)$$

Then, similar to ordinary Lyapunov balancing, a reduced order model can be obtained by balancing the pair of positive definite matrices (P_g, Q_g) and truncating based on the so-called generalized Hankel singular values (GHSV) which are the square roots of the eigenvalues of the product $P_g Q_g$. Then, similarly, the corresponding model reduction error bound is twice the sum of the neglected GHSV. For details see [16, Section 4.7].

As mentioned before, after applying balancing transformations, the relevant gramians are equal and diagonal. In case the state transformation only makes the gramians diagonal, but not necessarily equal, we say that the system is *essentially balanced*. It is easy to observe that, with the same truncation decision, the reduced order model obtained from balancing is equal to the one obtained from essentially balancing; that is, the transfer matrix of the reduced order model (from u to y) will be the same in both cases.

Let G denote the transfer matrix from u to y in system (5), i.e. $G(s) = C(sI - A)^{-1}B + D$. Then we call the linear system (5) *bounded real* if

$$G^\top(-j\omega)G(j\omega) \leq I, \quad \forall \omega \in \mathbb{R}. \quad (7)$$

We call the system (5) *strictly bounded real* if the above inequality is strict. Under the assumption that $I - D^\top D$ is nonsingular, strict bounded realness of G is equivalent to the \mathcal{H}_∞ -norm of G being strictly less than 1. If $I - D^\top D > 0$, then (5) is bounded real if and only if there exists a real symmetric matrix K satisfying the Riccati equation

$$A^\top K + KA + C^\top C + (KB + C^\top D)(I - D^\top D)^{-1}(KB + C^\top D)^\top = 0. \quad (8)$$

In fact in that case, all real symmetric solutions of (8) are positive definite and any real symmetric solution K lies between two extremal solutions K_m and K_M , that is $0 < K_m \leq K \leq K_M$. Bounded real balancing involves balancing the pair of positive definite matrices (K_M^{-1}, K_m) , and truncating based on the so-called bounded real characteristic values which are the square root of the eigenvalues of the product $K_M^{-1}K_m$. The error bound for Lyapunov balancing also holds for BR model reduction by considering neglected bounded real characteristic values instead of neglected HSV. Moreover, minimality, stability, and strict bounded realness are preserved in the reduced order model (see [17,13,12] for more details).

3. Stability preserving model reduction

In this section, we assume that agents have stable internal dynamics and network (2) is stable. Then, we reduce the dynamic order of the agents such that stability is preserved in the reduced order network.

3.1. Stability of the network

First, we analyze the stability of network (2). As the Laplacian matrix L is symmetric, there exists an orthogonal matrix U such that $U^\top L U = \Lambda$, where Λ is a diagonal matrix having the eigenvalues of L as diagonal elements, and the columns of U

are corresponding eigenvectors of L . Since L is positive semi-definite, and its row sums are zero, we can assume, without loss of generality, that

$$\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_p) \quad (9a)$$

with

$$0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_p \quad (9b)$$

and the first column of U is the normalized vector of ones. By applying the state space transformation $\tilde{x} = (U^\top \otimes I)x$ to (2), we obtain

$$\dot{\tilde{x}}(t) = (I \otimes A - \Lambda \otimes BC)\tilde{x}(t). \quad (10)$$

Observe that

$$\begin{aligned} I \otimes A - \Lambda \otimes BC \\ = \text{blockdiag}(A - \lambda_1 BC, A - \lambda_2 BC, \dots, A - \lambda_p BC). \end{aligned} \quad (11)$$

Hence, the linear system (2) is stable if and only if $A - \lambda_i BC$ is Hurwitz for each $i = 1, 2, \dots, p$.

The above necessary and sufficient stability condition cannot be directly applied to our model reduction framework. Besides, it requires information on the exact location of all eigenvalues of the Laplacian matrix which in some cases may not be available. Instead, we are interested in deducing stability of (2) by a small gain type of argument. Note that internal stability of the agents, i.e. A being Hurwitz, is necessary for stability of network (2). Assuming A to be Hurwitz and denoting the transfer matrix from u_i to y_i by G , we seek for a condition on the \mathcal{H}_∞ -norm of G under which the network (2) is stable. This brings us to the following lemma.

Lemma 3.1. Assume that A is Hurwitz. Let G denote the transfer matrix from u_i to y_i in (1), i.e. $G(s) = C(sI - A)^{-1}B$. Then, the network (2) is stable if

$$\lambda_p \|G\|_\infty < 1. \quad (12)$$

Proof. Assume that (12) holds. Then there exists a positive definite solution $X > 0$ to the inequality (see [18])

$$A^\top X + XA + C^\top C + \lambda_p^2 XBB^\top X < 0. \quad (13)$$

For any $\alpha \in \{\lambda_1, \lambda_2, \dots, \lambda_p\}$, we have

$$\begin{aligned} (A - \alpha BC)^\top X + X(A - \alpha BC) \\ = A^\top X + XA - \alpha(C^\top B^\top X + XBC) \\ = A^\top X + XA + C^\top C + \alpha^2 XBB^\top X \\ - (\alpha XB + C^\top)(\alpha B^\top X + C) \\ < (\alpha^2 - \lambda_p^2)XBB^\top X - (\alpha XB + C^\top)(\alpha B^\top X + C) \end{aligned}$$

where (13) is used to obtain the last inequality. Hence, $(A - \alpha BC)^\top X + X(A - \alpha BC)$ is negative definite for all $\alpha \in \{\lambda_1, \lambda_2, \dots, \lambda_p\}$. Consequently, $A - \lambda_i BC$ is Hurwitz for each $i = 1, 2, \dots, p$ and the network (2) is stable. \square

Remark 3.2. The value of λ_p plays a crucial role in the feasibility of the condition (12). Clearly, this condition tends to be more restrictive as the size of the network increases. Besides, the value of λ_p is bounded from below by the maximal degree of the vertices plus 1 (see [19]). Consequently, (12) is more likely to be satisfied in graphs like path or cycle graphs rather than star or complete graphs.

3.2. Model reduction

The result of Lemma 3.1 can be used to obtain a model reduction technique which preserves stability of the multi-agent system (1). Let $\Gamma(A, B, C)$ denote the linear system (1a) representing the dynamics of the individual agents. Assume that A is Hurwitz and the small gain condition (12) holds. Then, clearly, the linear system $\Gamma(A, \lambda_p B, C)$ is strictly bounded real. Therefore, bounded real balancing can be applied based on the Riccati equation

$$A^\top K + KA + C^\top C + \lambda_p^2 KBB^\top K = 0. \quad (14)$$

Let K_m and K_M denote the minimal and maximal real symmetric solutions of (14), respectively. Then, $0 < K_m \leq K_M$. In balanced coordinates we have, $K_m = K_M^{-1} = \Sigma$ where

$$\Sigma = \text{diag}\{\sigma_1 I_{s_1}, \sigma_2 I_{s_2}, \dots, \sigma_N I_{s_N}\} \quad (15)$$

with distinct bounded real characteristic values σ_i ordered in a decreasing manner. Consequently, for each positive integer $1 \leq k < N$, one can obtain a reduced order model $\Gamma_r(\bar{A}, \lambda_p \bar{B}, \bar{C})$ of order $r = \sum_{i=1}^k s_i$ by truncating the state components corresponding to the $(N-k)$ smallest distinct characteristic values. Then, obviously, the reduced agents' model $\Gamma_r(\bar{A}, \bar{B}, \bar{C})$ can be retrieved from Γ_r . This results in the following reduced order network:

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) \quad (16)$$

where $\bar{A} := I_p \otimes \bar{A} - L \otimes \bar{B}\bar{C}$ and $\bar{x} = \text{col}(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p)$ with $\bar{x}_i \in \mathbb{R}^r$ being the reduced state component of agent i for $i = 1, 2, \dots, p$. The reduced order network obtained in this way is stable as stated in the following theorem.

Theorem 3.3. Consider the multi-agent system (1), and assume that the small gain condition (12) holds. Let K_m and K_M denote the minimal and maximal real symmetric solutions of the Riccati equation (14). Then, for each positive integer $1 \leq k < N$, the reduced order network (16) of order pr with $r = \sum_{i=1}^k s_i$, obtained by balancing (K_M^{-1}, K_m) and truncating according to (15) is stable.

Proof. As observed earlier, balancing (K_M^{-1}, K_m) in (14) corresponds to BR balancing of $\Gamma(A, \lambda_p B, C)$. Therefore, the reduced order dynamics $\Gamma_r(\bar{A}, \lambda_p \bar{B}, \bar{C})$ is stable and strictly bounded real. Therefore, $\lambda_p \|G_r\|_\infty < 1$ where $G_r(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B}$. Hence, the reduced network (16) is stable by Lemma 3.1. \square

3.3. Error bounds

Assuming that stability of the original network holds due to the small gain condition (12), the model reduction scheme established in the previous subsection obtains reduced order dynamics for the agents in such a way that stability of the network is preserved in the reduced model (16). Moreover, one can show that the reduced order agent dynamics is close to the original one by establishing error bounds as follows.

As observed earlier, balancing the pair (K_M^{-1}, K_m) in Riccati equation (14) is equivalent to bounded real balancing with respect to the linear system $\Gamma(A, \lambda_p B, C)$. Hence, a model reduction error bound for the individual agents is obtained as

$$\|G - G_r\|_\infty \leq \frac{2}{\lambda_p} \sum_{i=r+1}^N \sigma_i \quad (17)$$

where G and G_r are the transfer matrices corresponding to $\Gamma(A, B, C)$ and $\Gamma_r(\bar{A}, \bar{B}, \bar{C})$, respectively. Note that to write (17), we have assumed that λ_p is nonzero, meaning that the graph has at least one edge. Obviously, the model reduction problem for the case where $\lambda_p = 0$ is not of current interest as it boils down to the ordinary model reduction problem of finite-dimensional linear time-invariant systems.

4. Synchronization preserving model reduction

4.1. Synchronization of the network

A synchronized network has the property that the state trajectories of the coupled agents converge to a common trajectory. More precisely, we have the following definition.

Definition 4.1. The multi-agent system (2) is *synchronized* if every solution of (2) satisfies $\lim_{t \rightarrow \infty} (x_j(t) - x_k(t)) = 0$ for all $j, k = 1, 2, \dots, p$.

Thus, different from network stability, network synchronization requires that the differences of the states of the agents converge to zero as time runs off to infinity.

In contrast with the previous section, here agents are allowed to, and typically have unstable dynamics. Therefore, throughout this section we assume that the underlying graph G is connected which is necessary for achieving synchronization in case of unstable agents' dynamics (see [5,1]).

Let the state disagreement vector z be defined as

$$z(t) = (R^\top \otimes I_n)x(t) \quad (18)$$

where $z = \text{col}(z_1, z_2, \dots, z_{p-1})$ and R is the incidence matrix of the graph. Then it is easy to observe that the network (2) is synchronized if and only if $\lim_{t \rightarrow \infty} z(t) = 0$ for any solution x of (2). Consequently, the network (2) is synchronized if and only if the system (2) with output variable z is output stable. Necessary and sufficient conditions for synchronization have already been investigated in the literature (see e.g. [5,20]). These conditions and the corresponding proofs are recapped in the following lemma for later use.

Lemma 4.2. Assume that the graph $G = (V, E)$ is connected, and let the eigenvalues of the Laplacian matrix be given by (9b). Then the network (2) is synchronized if and only if $A - \lambda_i BC$ is Hurwitz for each $i = 2, 3, \dots, p$.

Proof. The Laplacian matrix L admits the spectral decomposition $U^\top L U = \Lambda$ where Λ is given by (9a) and U is an orthogonal matrix with its first column being the normalized vector of ones. By applying the state space transformation $\tilde{x} = (U^\top \otimes I)x$ to system (2) with output Eq. (18), we obtain

$$\dot{\tilde{x}}(t) = (I_p \otimes A - \Lambda \otimes BC)\tilde{x}(t) \quad (19a)$$

$$z(t) = (R^\top U \otimes I_n)x(t). \quad (19b)$$

Observe that

$$\begin{aligned} I_p \otimes A - \Lambda \otimes BC \\ = \text{blockdiag}(A, A - \lambda_2 BC, \dots, A - \lambda_p BC) \end{aligned} \quad (20)$$

and $R^\top U \otimes I_n$ can be decomposed as

$$R^\top U \otimes I_n = \begin{bmatrix} 0 & R^\top \tilde{U} \otimes I_n \end{bmatrix} \quad (21)$$

where \tilde{U} is the matrix obtained by deleting the first column of U , and 0 is a zero matrix with n columns. For any initial state $\tilde{x}(0) = x_0$, the output of the system (19) is obtained as

$$z(t) = (R^\top U \otimes I_n)e^{((I \otimes A) - (\Lambda \otimes BC))t} x_0$$

which, by using (20) and (21), can be rewritten as

$$\begin{aligned} z(t) = \begin{bmatrix} 0 & R^\top \tilde{U} \otimes I_n \end{bmatrix} \\ \times \text{blockdiag}(e^{At}, e^{(A - \lambda_2 BC)t}, \dots, e^{(A - \lambda_p BC)t}) x_0. \end{aligned}$$

Since R is the incidence matrix associated with a connected graph, the matrix $R^\top \tilde{U} \otimes I_n$ has full column rank. Thus, the output variable z converges to zero for any x_0 if and only if $A - \lambda_i BC$ is Hurwitz for each $i = 2, 3, \dots, p$. \square

4.2. Model reduction

Starting from a synchronized network, our aim here is to derive a reduced order model for the network such that synchronization is preserved in this reduced order model. Recall that the model reduction technique proposed in the previous section uses the scaled dynamics of the agents $\Gamma(A, \lambda_p B, C)$ to obtain a lower order network (16). In the context of synchronization, however, the individual agents' dynamics is not necessarily stable. Therefore, usual balancing methods like Lyapunov balancing and BR balancing cannot be applied directly to the original agents' dynamics in this case. The idea here is to use stable dynamics present in a synchronized network; in particular, $A - \lambda_i BC$ with $i = 2, 3, \dots, p$ (see Lemma 4.2). In terms of these dynamics, small gain type of conditions can also be derived to guarantee synchronization of (2) as stated in the following lemma.

Lemma 4.3. *Let the eigenvalues of the Laplacian matrix be given by (9b). Then the network (2) is synchronized if there exists an index $\ell \in \{2, 3, \dots, p\}$ such that $A - \lambda_\ell BC$ is Hurwitz and*

$$\delta \|H_\ell\|_\infty < 1 \quad (22)$$

where

$$H_\ell(s) = C(sI - A + \lambda_\ell BC)^{-1}B$$

and

$$\delta = \max(\lambda_\ell - \lambda_2, \lambda_p - \lambda_\ell). \quad (23)$$

Proof. Suppose that (22) holds. Then there exists a positive definite matrix X which satisfies the Riccati inequality (see [18])

$$(A - \lambda BC)^T X + X(A - \lambda BC) + C^T C + \delta^2 X B B^T X < 0 \quad (24)$$

where the index ℓ is dropped for notational convenience. For each $i = 2, 3, \dots, p$, we have

$$\begin{aligned} (A - \lambda_i BC)^T X + X(A - \lambda_i BC) &= (A - \lambda BC)^T X + X(A - \lambda BC) \\ &\quad - (\lambda_i - \lambda)(C^T B^T X + X B C) \\ &= (A - \lambda BC)^T X + X(A - \lambda BC) + C^T C + \delta^2 X B B^T X \\ &\quad + ((\lambda_i - \lambda)^2 - \delta^2) X B B^T X \\ &\quad - ((\lambda_i - \lambda) X B + C^T)((\lambda_i - \lambda) B^T X + C). \end{aligned} \quad (25)$$

Now, the right hand side is negative definite due to (23) and (24). Therefore $A - \lambda_i BC$ is Hurwitz for each $i = 2, 3, \dots, p$, and the network is synchronized by Lemma 4.2. \square

Remark 4.4. As mentioned before, in the context of synchronization, agents typically have unstable dynamics, i.e. A is not Hurwitz. Based on the proof of Lemma 4.3, it can be shown that in this case, the condition (22) is satisfied only if $2\lambda_\ell > \lambda_p$. To see this, suppose (22) holds, and consider (25) where λ_i is replaced by zero. Then, since A is not Hurwitz, the right hand side of (25) cannot be negative definite. This results in $\lambda_\ell > \delta$ which yields $2\lambda_\ell > \lambda_p$. Consequently, searching for λ_ℓ in Lemma 4.3 can be restricted, in this generic case, to the interval $\frac{\lambda_p}{2} < \lambda_\ell \leq \lambda_p$.

Remark 4.5. The feasibility of the condition (22) depends both on the dynamics of the agents and the magnitude of δ . Hence, for given agent dynamics, the so-called Laplacian spread of a graph, given by $\lambda_p - \lambda_2$, plays a crucial role in the feasibility of (22). Consequently, the condition (22) is expected to be more restrictive for graphs with a large Laplacian spread like star graphs, and to be less restrictive

as the underlying graph tends to a complete graph. For details regarding the Laplacian spread of a graph see [21,22].

The theorem above can be used for model reduction purposes. Starting from the assumption that the condition (22) holds for the original network, implying that the original network is synchronized, we obtain a reduced order model such that synchronization is preserved in the reduced model. This is illustrated next.

Assume that synchronization of network (2) is verified by condition (22). In addition, suppose that $\delta \neq 0$, i.e. the underlying communication topology does not corresponds to a complete graph. Model reduction for the case $\delta = 0$ is rather trivial and will be discussed later (see Remark 4.9). As before, let λ_ℓ be denoted shortly by λ . Choose $\gamma > 0$ such that

$$\delta \|C(sI - A + \lambda BC)^{-1}B\|_\infty < \gamma < 1. \quad (26)$$

Then there exists a positive definite matrix K satisfying the Riccati equation

$$(A - \lambda BC)^T K + K(A - \lambda BC) + C^T C + \left(\frac{\delta}{\gamma}\right)^2 K B B^T K = 0. \quad (27)$$

Observe that $\Gamma(A - \lambda BC, \frac{\delta}{\gamma} B, C)$ is strictly bounded real for any choice of γ satisfying (26). Let K_m and K_M denote the minimal and maximal real symmetric solutions of (27). Then BR balancing can be applied using the pair of positive definite matrices (K_M^{-1}, K_m) in order to obtain a reduced order model Γ_r from the original dynamics Γ . Obviously, the reduced order dynamics of the agents, $\Gamma_r(\bar{A}, \bar{B}, \bar{C})$, can be then retrieved from Γ_r . Consequently, a reduced order network is obtained which can be written again as in (16). Moreover, synchronization is preserved in the reduced order model as stated in the following theorem.

Theorem 4.6. *Consider the network (2), and assume that the condition (22) holds. Let K_m and K_M denote the minimal and maximal real symmetric solutions of the Riccati equation (27). Then, for each positive integer $1 \leq k < N$, the reduced order network (16) of order pr with $r = \sum_{i=1}^k s_i$, obtained by balancing (K_M^{-1}, K_m) and truncating according to (15) is synchronized.*

Proof. Following the discussion preceding the theorem, if (22) holds then the linear system $\Gamma(A - \lambda BC, \frac{\delta}{\gamma} B, C)$ is strictly bounded real for any γ satisfying (26). Therefore, due to the properties of BR balancing, the reduced system $\Gamma_r(\bar{A} - \lambda \bar{B} \bar{C}, \frac{\delta}{\gamma} \bar{B}, \bar{C})$ obtained by balancing (K_M^{-1}, K_m) and truncating is stable and bounded real. Then, since $\gamma < 1$, we have

$$\|\delta C(sI - \bar{A} + \lambda \bar{B} \bar{C})^{-1} \bar{B}\|_\infty < 1$$

and the reduced order network (16) is synchronized by Lemma 4.3. \square

4.3. Error bound

Assuming that the small gain condition (22) holds, the model reduction scenario proposed in the previous section obtains a reduced order network which preserves synchronization. Here, we show that the reduced order network also gives a good approximation of the behavior of the original network.

As mentioned earlier, in the context of synchronization agents typically have unstable dynamics, which makes it difficult to compare the output of the original dynamics of the agent to that of the reduced one. However, as observed, not the agent's state components but their difference plays a crucial role in synchronization. Therefore, to establish model reduction error bounds we look at the differences of the outputs of the agents, and

we do so for each pair of agents which communicate with each other. More precisely, we define the *output disagreement vector* as

$$\xi(t) := (R^\top \otimes I_n)y(t)$$

where $y(t) = \text{col}(y_1, y_2, \dots, y_p)$ and R is the incidence matrix. This can be rewritten as

$$\xi(t) = (R^\top \otimes C)x(t). \quad (28)$$

Furthermore, as the network (2) is an autonomous system, we also need to introduce an auxiliary input in order to be able to compare the input–output behavior of the original network to that of the reduced order model. Hence, we add a disturbance term d_i in the diffusively coupled feedback law (1b). That is, we replace (1b) by

$$\tilde{u}_i(t) = - \sum_{(i,j) \in E} (y_i(t) - y_j(t)) + d_i(t). \quad (29)$$

Consequently, we obtain the compact form

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}x(t) + (I_p \otimes B)d(t) \\ \xi(t) &= (R^\top \otimes C)x(t) \end{aligned} \quad (30)$$

where $d = \text{col}(d_1, d_2, \dots, d_p)$ and $\mathcal{A} := I_p \otimes A - L \otimes BC$. Then, for the reduced r th order dynamics we have

$$\begin{aligned} \dot{\tilde{x}}(t) &= \tilde{\mathcal{A}}\tilde{x}(t) + (I_p \otimes \tilde{B})d(t) \\ \xi(t) &= (R^\top \otimes \tilde{C})\tilde{x}(t) \end{aligned} \quad (31)$$

with $\tilde{x}_i \in \mathbb{R}^r$, $\tilde{x} = \text{col}(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_p)$ and $\tilde{\mathcal{A}} := I_p \otimes \tilde{A} - L \otimes \tilde{B}\tilde{C}$.

Now, let T and T_r denote the transfer matrices from d to ξ in (30) and (31), respectively. Then $\|T - T_r\|_\infty$ measures the model reduction error, and we have the following result.

Theorem 4.7. Consider the network (2) and assume that there exist an ℓ and a nonzero δ such that the condition (22) holds. Let γ be a parameter satisfying (26). Let K_m and K_M denote the minimal and maximal real symmetric solutions of the Riccati equation (27). Let (31) represent the reduced order network obtained by balancing (K_M^{-1}, K_m) and truncating according to (15). Then, we have the following model reduction error bound:

$$\|T - T_r\|_\infty \leq \frac{2\gamma\sqrt{\lambda_p}}{\delta(1-\gamma^2)} \sum_{i=r+1}^N \sigma_i, \quad (32)$$

where the σ_i s are the diagonal elements of Σ in (15).

Proof. The Laplacian matrix L admits the following spectral decomposition

$$U^\top LU = \Lambda \quad (33)$$

where Λ is given by (9), and the first column of U is the normalized vector of ones. By applying the state space transformation $\tilde{x} = (U^\top \otimes I_n)x$ to (30) we obtain

$$\begin{aligned} \dot{\tilde{x}}(t) &= (I_p \otimes A - \Lambda \otimes BC)\tilde{x}(t) + (U^\top \otimes B)d(t) \\ \xi(t) &= (R^\top U \otimes C)\tilde{x}(t). \end{aligned} \quad (34)$$

Observe that

$$I_p \otimes A - \Lambda \otimes BC = \text{blockdiag}(A, A - \lambda_2 BC, \dots, A - \lambda_p BC),$$

and $R^\top U \otimes C$ is of the form

$$R^\top U \otimes C = \begin{bmatrix} 0 & R^\top \tilde{U} \otimes C \end{bmatrix}$$

where \tilde{U} is the matrix obtained by deleting the first column of U , and 0 is a zero matrix with n columns. Let \tilde{x} be partitioned

accordingly as $\tilde{x} = \text{col}(\tilde{x}_1, \tilde{x}_2)$ where $\tilde{x}_1 \in \mathbb{R}^n$. Then, the network Eq. (34) simplifies to

$$\begin{aligned} \dot{\tilde{x}}_2(t) &= (I_{p-1} \otimes A - \tilde{\Lambda} \otimes BC)\tilde{x}_2(t) + (\tilde{U}^\top \otimes B)d(t) \\ \xi(t) &= (R^\top \tilde{U} \otimes C)\tilde{x}_2(t) \end{aligned} \quad (35)$$

where $\tilde{\Lambda} = \text{diag}(\lambda_2, \dots, \lambda_p)$. Note that \tilde{x}_1 does not appear in the above as it corresponds to unobservable state variables.

Analogously, for the reduced order network (31) we obtain the following state space representation:

$$\begin{aligned} \dot{\hat{x}}_2(t) &= (I_{p-1} \otimes \tilde{A} - \tilde{\Lambda} \otimes \tilde{B}\tilde{C})\hat{x}_2(t) + (\tilde{U}^\top \otimes \tilde{B})d(t) \\ \xi(t) &= (R^\top \tilde{U} \otimes \tilde{C})\hat{x}_2(t). \end{aligned} \quad (36)$$

Clearly, the transfer matrix from d to ξ in (35) is equal to that of (30), i.e. to T , and the transfer matrix from d to ξ in (36) is equal to that of (31), i.e. to T_r .

Now, we write the Lyapunov inequalities (6) for system (35) as

$$\begin{aligned} (I_{p-1} \otimes A - \tilde{\Lambda} \otimes BC)^\top Q_g + Q_g (I_{p-1} \otimes A - \tilde{\Lambda} \otimes BC) \\ + (R^\top \tilde{U} \otimes C)^\top (R^\top \tilde{U} \otimes C) &\leq 0 \\ (I_{p-1} \otimes A - \tilde{\Lambda} \otimes BC)P_g + P_g (I_{p-1} \otimes A - \tilde{\Lambda} \otimes BC)^\top \\ + (\tilde{U}^\top \otimes B)(\tilde{U}^\top \otimes B)^\top &\leq 0 \end{aligned}$$

which simplifies to

$$\begin{aligned} (I_{p-1} \otimes A - \tilde{\Lambda} \otimes BC)^\top Q_g + Q_g (I_{p-1} \otimes A - \tilde{\Lambda} \otimes BC) \\ + \tilde{\Lambda} \otimes C^\top C &\leq 0 \end{aligned} \quad (37a)$$

$$\begin{aligned} (I_{p-1} \otimes A - \tilde{\Lambda} \otimes BC)P_g + P_g (I_{p-1} \otimes A - \tilde{\Lambda} \otimes BC)^\top \\ + I_{p-1} \otimes BB^\top &\leq 0. \end{aligned} \quad (37b)$$

Note that equality (4) and the fact that $\tilde{U}^\top L \tilde{U} = \tilde{\Lambda}$ are used to write (37a). Now, assume that there exist an ℓ and a nonzero δ such that (22) holds. Let again λ_ℓ be denoted shortly by λ , and recall that K_m and K_M are the minimal and maximal real symmetric solutions of (27), respectively. Then we claim that

$$Q_g = \frac{\lambda_p}{1-\gamma^2} (I \otimes K_m) \quad (38a)$$

and

$$P_g = \frac{\gamma^2}{\delta^2(1-\gamma^2)} (I \otimes K_M^{-1}) \quad (38b)$$

satisfy the Lyapunov inequalities (37a) and (37b), respectively.

It is easy to observe that Q_g given by (38a) satisfies (37a) if and only if

$$(A - \lambda_i BC)^\top K_m + K_m (A - \lambda_i BC) + \frac{\lambda_i(1-\gamma^2)}{\lambda_p} C^\top C \leq 0$$

for each $i = 2, 3, \dots, p$. We have

$$\begin{aligned} (A - \lambda_i BC)^\top K_m + K_m (A - \lambda_i BC) + \frac{\lambda_i(1-\gamma^2)}{\lambda_p} C^\top C \\ = (A - \lambda BC)^\top K_m + K_m (A - \lambda BC) \\ - (\lambda_i - \lambda)(C^\top B^\top K_m + K_m BC) + \frac{\lambda_i(1-\gamma^2)}{\lambda_p} C^\top C \\ = -\frac{\delta^2}{\gamma^2} K_m B B^\top K_m - \left(1 - \frac{\lambda_i(1-\gamma^2)}{\lambda_p}\right) C^\top C \\ - (\lambda_i - \lambda)(C^\top B^\top K_m + K_m BC) \\ = -[K_m B \quad C] \begin{bmatrix} \frac{\delta^2}{\gamma^2} I & (\lambda_i - \lambda)I \\ (\lambda_i - \lambda)I & \left(1 - \frac{\lambda_i(1-\gamma^2)}{\lambda_p}\right) I \end{bmatrix} \begin{bmatrix} B^\top K_m \\ C \end{bmatrix} \end{aligned}$$

where (27) is used to derive the second equality. Therefore, Q_g given by (38a) satisfies (37a) if

$$\begin{bmatrix} \frac{\delta^2}{\gamma^2} & \lambda_i - \lambda \\ \lambda_i - \lambda & 1 - \frac{\lambda_i(1 - \gamma^2)}{\lambda_p} \end{bmatrix} \geq 0 \quad (39)$$

for $i = 2, 3, \dots, p$. This holds if and only if

$$\frac{\lambda_i(1 - \gamma^2)}{\lambda_p} + \frac{\gamma^2}{\delta^2}(\lambda_i - \lambda)^2 \leq 1. \quad (40)$$

Recall that, by definition, $\delta = \max(\lambda_p - \lambda, \lambda - \lambda_2)$. Hence, the first term on the left hand side of (40) is not greater than $(1 - \gamma^2)$, and the second term is not greater than γ^2 . Therefore, Q_g given by (38a) satisfies Lyapunov inequalities (37a).

Now, we will show that P_g given by (38b) satisfies (37b). Clearly this holds if and only if

$$(A - \lambda_i BC)K_M^{-1} + K_M^{-1}(A - \lambda_i BC)^\top + \frac{\delta^2(1 - \gamma^2)}{\gamma^2}BB^\top \leq 0 \quad (41)$$

for each $i = 2, 3, \dots, p$. By multiplying (41) from the left and right by K_M , we obtain

$$\begin{aligned} (A - \lambda_i BC)^\top K_M + K_M(A - \lambda_i BC) \\ + \frac{\delta^2(1 - \gamma^2)}{\gamma^2}K_M BB^\top K_M \leq 0. \end{aligned} \quad (42)$$

We have

$$\begin{aligned} (A - \lambda_i BC)^\top K_M + K_M(A - \lambda_i BC) \\ + \frac{\delta^2(1 - \gamma^2)}{\gamma^2}K_M BB^\top K_M \\ = (A - \lambda BC)^\top K_M + K_M(A - \lambda BC) \\ - (\lambda_i - \lambda)(C^\top B^\top K_M + K_M BC) \\ + \frac{\delta^2(1 - \gamma^2)}{\gamma^2}K_M BB^\top K_M \\ = -C^\top C - \delta^2 K_M BB^\top K_M \\ - (\lambda_i - \lambda)(C^\top B^\top K_M + K_M BC) \\ = -[K_M B \quad C^\top] \begin{bmatrix} \delta^2 I & (\lambda_i - \lambda)I \\ (\lambda_i - \lambda)I & I \end{bmatrix} \begin{bmatrix} B^\top K_M \\ C \end{bmatrix} \end{aligned}$$

where (27) is used to derive the second equality. Therefore, P_g given by (38b) satisfies (37b) if

$$\begin{bmatrix} \delta^2 & \lambda_i - \lambda \\ \lambda_i - \lambda & 1 \end{bmatrix} \geq 0 \quad (43)$$

for each $i = 2, 3, \dots, p$. This holds if and only if

$$\delta \geq |\lambda_i - \lambda|$$

which is true by the definition of δ . Hence, P_g given by (38b) satisfies Lyapunov inequalities (37b). Consequently, P_g and Q_g are generalized gramians for system (35).

Now, in the balanced coordinates, we have $K_m = K_M^{-1} = \Sigma$ where Σ is given by (15). Hence, after balancing, P_g and Q_g are obtained as

$$P_g = \frac{\gamma^2}{\delta^2(1 - \gamma^2)}(I \otimes \Sigma)$$

and

$$Q_g = \frac{\lambda_p}{1 - \gamma^2}(I \otimes \Sigma).$$

Thus, both P_g and Q_g will become diagonal after balancing (K_M^{-1}, K_m) . Therefore, balancing (K_M^{-1}, K_m) yields essentially balancing of the generalized gramians (P_g, Q_g) of the network, see Section 2.2. Note that the generalized Hankel singular values are the square roots of the eigenvalues of the product $P_g Q_g$ which in this case are the diagonal elements of the matrix $\frac{\gamma \sqrt{\lambda_p}}{\delta(1 - \gamma^2)} \Sigma$. This establishes the model reduction error bound (32). \square

Remark 4.8. Recall that the parameter γ is chosen such that (26) holds. Obviously, different choices of γ lead to different reduced order models. Although the error bound proposed in Theorem 4.7 is not optimal in any norm, heuristically, one can choose γ such that the guaranteed error bound in (32) is as small as possible. Note that the singular values σ_i also depend on γ .

Remark 4.9. In case where the communication topology corresponds to a complete graph, reducing dynamics from d to ξ in (30) boils down to an ordinary model reduction problem of a linear system (see e.g. [14, p. 28] for the Laplacian spectrum of complete graphs). In particular, (35) can be written as

$$\begin{aligned} \dot{\tilde{x}}_2(t) &= (I_{p-1} \otimes (A - pBC))\tilde{x}_2 + (\tilde{U} \otimes B)d(t) \\ \xi(t) &= (R^\top \tilde{U} \otimes C)\tilde{x}_2(t). \end{aligned}$$

Note that for a complete graph we have $\lambda_2 = \lambda_3 = \dots = \lambda_p = p$. Then, one can write the corresponding Lyapunov equations as

$$\begin{aligned} (I_{p-1} \otimes (A - pBC))^\top Q + Q(I_{p-1} \otimes (A - pBC)) \\ + pI_{p-1} \otimes C^\top C = 0 \\ (I_{p-1} \otimes (A - pBC))P + P(I_{p-1} \otimes (A - pBC))^\top \\ + I_{p-1} \otimes BB^\top = 0, \end{aligned}$$

which is simplified to

$$\begin{aligned} (A - pBC)^\top Q + Q(A - pBC) + pC^\top C = 0 \\ (A - pBC)P + P(A - pBC) + BB^\top = 0. \end{aligned}$$

Consequently, one can apply Lyapunov balanced truncation to the stable linear system $\Gamma(A - pBC, B, \sqrt{p}C)$, and obtain a reduced order network.

5. Numerical example

Here, we apply the proposed synchronization preserving model reduction approach established in the previous section to a numerical example. Consider the spacecraft formation problem studied in [5]. The dynamics of the individual agents is given by

$$\begin{bmatrix} \dot{r}_i \\ \dot{\ddot{r}}_i \end{bmatrix} = \begin{bmatrix} 0 & I_3 \\ A_1 & A_2 \end{bmatrix} \begin{bmatrix} r_i \\ \dot{r}_i \end{bmatrix} + \begin{bmatrix} 0 \\ I_3 \end{bmatrix} u_i \quad (44)$$

with

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 \times 10^{-6} & 0 \\ 0 & 0 & -10^{-6} \end{bmatrix} \\ A_2 &= \begin{bmatrix} 0 & 2 \times 10^{-3} & 0 \\ -2 \times 10^{-3} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

where $r_i \in \mathbb{R}^3$ is the position vector of the i th agent (satellite). Let A denote the overall state matrix in (44). Suppose that state information of the agents is transmitted over the network through the static state feedback C given by Li et al. [5, Example 4]

$$C = \begin{bmatrix} 0.6596 & -0.0013 & 0 & 1.9789 & 0 & 0 \\ 0.0013 & 0.6596 & 0 & 0 & 1.9789 & 0 \\ 0 & 0 & 0.6596 & 0 & 0 & 1.9789 \end{bmatrix},$$

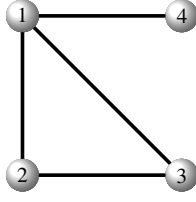


Fig. 1. Communication topology.

and the agents communicate according to the communication topology given by Fig. 1. Then, the network equations can be written in compact form as

$$\dot{x} = (I_4 \otimes A - L \otimes BC)x \quad (45)$$

where $x = \text{col}(r_1, \dot{r}_1, r_2, \dot{r}_2, r_3, \dot{r}_3, r_4, \dot{r}_4)$, $B = \text{col}(0, I_3)$, and the Laplacian matrix is given by

$$L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}. \quad (46)$$

The eigenvalues of the Laplacian matrix (46) are $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 3$, and $\lambda_4 = 4$. Observe that

$$(\lambda_4 - \lambda_2) \|C(sI - A + \lambda_4 BC)^{-1}B\|_\infty = 0.7746 < 1,$$

and, hence, (45) is synchronized by Lemma 4.3.

Now let $\gamma = 0.8$, which clearly satisfies (26). Let again K_m and K_M be the minimal and maximal real symmetric solutions of the corresponding Riccati (27). Then a balancing transformation for (K_M^{-1}, K_m) is computed as

$$T = \begin{bmatrix} 0.2194 & -0.0006 & 0.0000 & 0.7300 & -0.0002 & 0.0000 \\ -0.0006 & -0.2194 & 0.0000 & 0.0002 & -0.7300 & 0.0000 \\ 0.0000 & 0.0000 & 0.2194 & 0.0000 & 0.0000 & 0.7300 \\ -0.2194 & -0.0006 & 0.0000 & 0.0718 & 0.0003 & 0.0000 \\ -0.0006 & 0.2194 & 0.0000 & 0.0003 & -0.0718 & 0.0000 \\ 0.0000 & 0.0000 & -0.2194 & 0.0000 & 0.0000 & 0.0718 \end{bmatrix}, \quad (47)$$

and the corresponding diagonal matrix Σ is obtained as

$$\Sigma = \begin{bmatrix} 0.7827I_3 & 0 \\ 0 & 0.1927I_3 \end{bmatrix}.$$

Note that the only admissible truncation is to discard three state components. Now, by truncating the three state components corresponding to the smallest diagonal elements of Σ , the reduced order dynamics of the agents $\Gamma_r(\bar{A}, \bar{B}, \bar{C})$ is obtained as

$$\bar{A} = \begin{bmatrix} 0.2736 & -0.0012 & 0.0000 \\ 0.0012 & 0.2736 & 0.0000 \\ 0.0000 & 0.0000 & 0.2736 \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} 0.7300 & -0.0002 & 0.0000 \\ -0.0002 & -0.7300 & 0.0000 \\ 0.0000 & 0.0000 & 0.7300 \end{bmatrix},$$

and

$$\bar{C} = \begin{bmatrix} 2.7374 & -0.0010 & 0.0000 \\ -0.0010 & -2.7374 & 0.0000 \\ 0.0000 & 0.0000 & 2.7374 \end{bmatrix}.$$

Consequently, the corresponding reduced order network can be represented as

$$\dot{\bar{x}}(t) = (I_4 \otimes \bar{A} - L \otimes \bar{B}\bar{C})\bar{x}(t). \quad (48)$$

It is easy to verify that $\bar{A} - \lambda_i \bar{B}\bar{C}$ is Hurwitz for each $i \in \{2, 3, 4\}$, and, hence, the reduced order network is synchronized by Lemma 4.2. Alternatively, one can verify that

$$(\lambda_4 - \lambda_2) \|\bar{C}(sI - \bar{A} + \lambda_4 \bar{B}\bar{C})^{-1}\bar{B}\|_\infty = 0.7766 < 1,$$

which implies that (48) is synchronized by Lemma 4.3. Note that, in fact, while the static feedback C synchronizes the original agents' dynamics, the truncated matrix \bar{C} synchronizes the reduced order dynamics of the agents.

Now, to compare the behavior of the reduced order network to the original one, as in Section 4.3, we introduce auxiliary inputs and outputs to obtain the forms (30) and (31) for the original and the reduced order network, respectively. Note that in this example

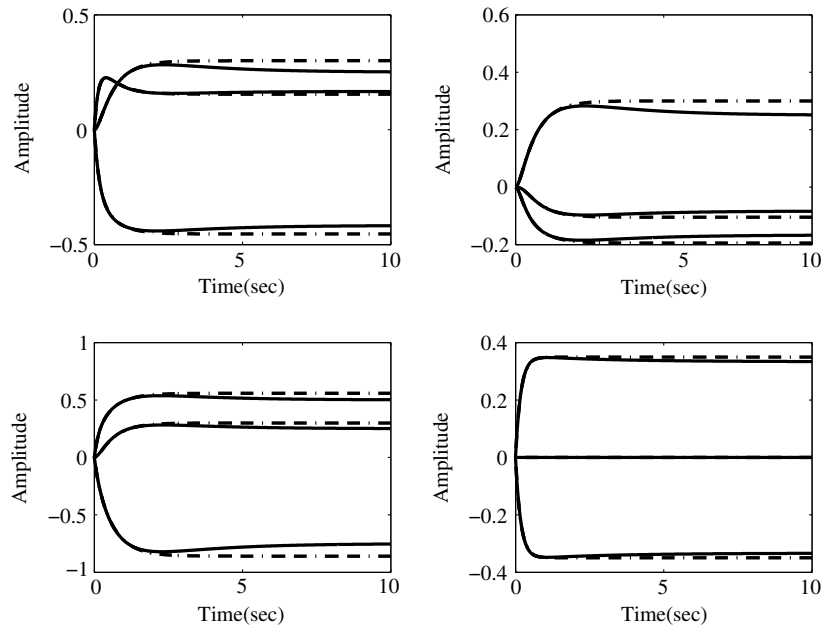


Fig. 2. Network model order reduction of Example 1: responses to the step disturbances corresponding to $\alpha_i = [1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0]$; original network (solid line) and reduced order network (dashed dotted line); ξ_1, ξ_2, ξ_3 (top left), ξ_4, ξ_5, ξ_6 (top right), ξ_7, ξ_8, ξ_9 (down left), $\xi_{10}, \xi_{11}, \xi_{12}$ (down right).

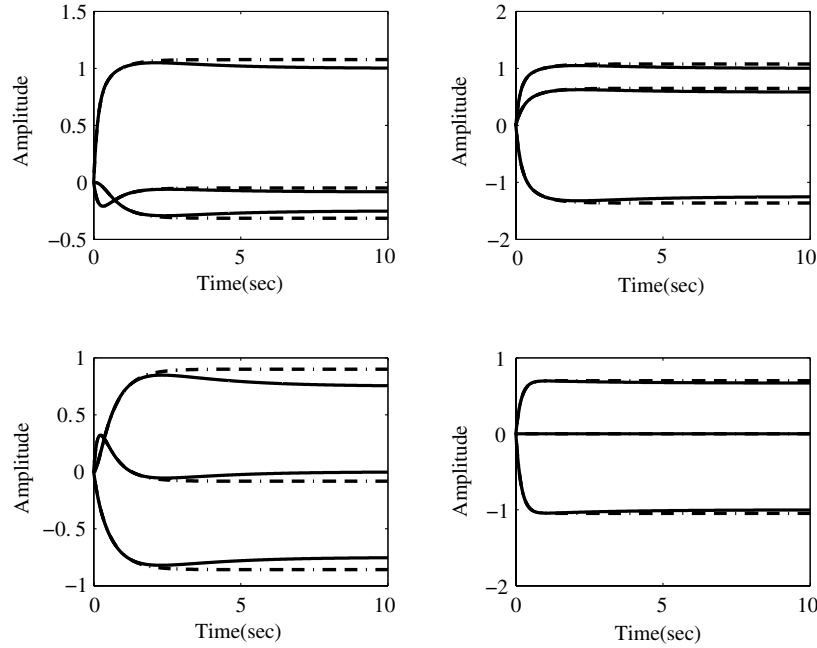


Fig. 3. Network model order reduction of Example 1: responses to the step disturbances corresponding to $\alpha_i = [2 \ 0 \ 3 \ 3 \ 0 \ 0 \ 1 \ 3 \ 0 \ 3 \ 0 \ 1]_i$; original network (solid line) and reduced order network (dashed dotted line); ξ_1, ξ_2, ξ_3 (top left), ξ_4, ξ_5, ξ_6 (top right), ξ_7, ξ_8, ξ_9 (down left), $\xi_{10}, \xi_{11}, \xi_{12}$ (down right).

we have $p = 4, n = 6, r = 3, \xi \in \mathbb{R}^{12}$, and the incidence matrix of the graph can be given as

$$R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Now, we apply step disturbances to some, randomly chosen, channels of (30) and (31). In particular, for each $i = 1, 2, \dots, 12$, we consider a disturbance $d_i = \alpha_i \mathbb{1}(t)$ where $\mathbb{1}(t)$ is the unit step function and α_i is some nonnegative integer indicating the amplitude of the step disturbance d_i . Clearly, the value of α_i is set to zero whenever the i th channel is not affected by the disturbance. For two different choices of α_i s, the output responses of (30) and (31), i.e. ξ variables, are compared in Figs. 2 and 3. It is also worth computing the actual model reduction error and the proposed error bound which corresponds to the left and right hand side of (32), respectively. In this case, the actual model reduction error is obtained as 0.1587 and the error bound is computed as 0.5674.

6. Conclusions

In this paper, we have studied the problem of model order reduction for multi-agent systems. The identical agents are assumed to be general finite dimensional linear time-invariant systems. Small gain types of condition were derived to guarantee stability and synchronization of networks. It was observed that these conditions depend both on the communication topology and dynamics of the agents. Two different scenarios for model reduction of multi-agent systems were considered. First, assuming that the agents have stable internal dynamics, and the overall network is stable, a stability preserving model reduction approach was established. Consequently, model reduction error bounds on the dynamics of the individual agents were obtained. In the second scheme, we started off by the assumption that the original network is synchronized and a certain small gain condition holds. Then, a synchronization preserving model reduction technique was proposed by using bounded real balancing of some network dynamics. After adding appropriate auxiliary inputs and outputs to

the initial network representation, the behavior of the original and the reduced order network were compared by establishing model reduction error bounds. The proposed model reduction scheme was applied to a numerical example. The simulation results shows that the reduced order network gives a good approximation of the original one, providing that the neglected bounded real characteristic values are relatively small.

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